

# Multiple Lattice Packings and Coverings of the Plane with Triangles

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## Abstract

Given a convex disk  $K$  and a positive integer  $j$ , let  $\delta_L^j(K)$  and  $\vartheta_L^j(K)$  denote the  $j$ -fold lattice packing density and the  $j$ -fold lattice covering density of  $K$ , respectively. I will prove that for every triangle  $T$  we have that  $\delta_L^j(T) = \frac{2j^2}{2j+1}$  and  $\vartheta_L^j(T) = \frac{2j+1}{2}$ . Furthermore, I also obtain that the numbers of lattices which attain these densities both are  $(2j+1) \prod_{p|2j+1} \left(1 - \frac{2}{p}\right)$ , where the product is over the distinct prime numbers dividing  $2j+1$ .

## 1 Introduction

Let  $S$  be a subset of  $\mathbb{R}^2$ . The measure of  $S$  will be denoted by  $|S|$ . The closure and the interior of  $S$  are denoted by  $\bar{S}$  and  $Int(S)$ , respectively. The cardinality of  $S$  is denoted by  $card\{S\}$ .

Let  $D$  be a measurable connected subset in  $\mathbb{R}^2$ . The *upper* and *lower density* of a family  $\mathcal{F} = \{K_1, K_2, \dots\}$  of measurable bounded sets with respect to  $D$  are defined as

$$d_+(\mathcal{F}, D) = \frac{1}{|D|} \sum_{K \in \mathcal{F}, K \cap D \neq \emptyset} |K|,$$

and

$$d_-(\mathcal{F}, D) = \frac{1}{|D|} \sum_{K \in \mathcal{F}, K \subset D} |K|.$$

We define the *upper* and *lower density* of the family  $\mathcal{F}$  by

$$d_+(\mathcal{F}) = \limsup_{I \rightarrow \infty} d_+(\mathcal{F}, I^2),$$

and

$$d_-(\mathcal{F}) = \liminf_{I \rightarrow \infty} d_-(\mathcal{F}, I^2),$$

where  $I = [-1, 1]$ .

A family of measurable bounded sets  $\mathcal{F} = \{K_1, K_2, \dots\}$  is said to be a  *$j$ -fold packing* of a connected set  $D$  provided  $\bigcup_i K_i \subset D$  and each point of  $D$

belongs to the interiors of at most  $j$  sets of the family. In particular, if  $D$  is the whole plane  $\mathbb{R}^2$ , then when all  $K_i$  are congruent to a fixed measurable bounded set  $K$  the corresponding family is called a *j-fold congruent packing* of  $\mathbb{R}^2$  with  $K$ , when all  $K_i$  are translates of  $K$  the corresponding family is called a *j-fold translative packing* of  $\mathbb{R}^2$  with  $K$ , and when the translative vectors form a lattice the corresponding family is called a *j-fold lattice packing* of  $\mathbb{R}^2$  with  $K$ . We define

$$\delta^j(K) = \sup_{\mathcal{F}} d_+(\mathcal{F}),$$

the supremum being taken over all  $j$  fold congruent packings  $\mathcal{F}$  of  $\mathbb{R}^2$  with  $K$ . Similarly, we can also define  $\delta_T^j(K)$  and  $\delta_L^j(K)$  for the  $j$ -fold translative packings and the  $j$ -fold lattice packings, respectively. Obviously, we have

$$\delta_L^j(K) = \max_{\Lambda} \frac{|K|}{d(\Lambda)}, \quad (1)$$

the maximum is over all lattices  $\Lambda$  such that  $K + \Lambda$  is a  $j$ -fold lattice packing of  $\mathbb{R}^2$ .

As a counterpart to a  $j$ -fold packing, a family of measurable bounded sets  $\mathcal{F} = \{K_1, K_2, \dots\}$  is said to be a *j-fold covering* of a connected set  $D$  if each point of  $D$  belongs to at least  $j$  sets of the family. Similar to the case of the packing, for a fixed measurable bounded set  $K$  we can define a *j-fold congruent covering*, a *j-fold translative covering* and a *j-fold lattice covering* of  $\mathbb{R}^2$  with  $K$ . We define

$$\vartheta^j(K) = \inf_{\mathcal{F}} d_-(\mathcal{F}),$$

the infimum being taken over all  $j$ -fold congruent coverings  $\mathcal{F}$  of  $\mathbb{R}^2$  with  $K$ . Similarly, we can define  $\vartheta_T^j(K)$  and  $\vartheta_L^j(K)$  for the  $j$ -fold translative coverings and the  $j$ -fold lattice coverings, respectively. Clearly, we have

$$\vartheta_L^j(K) = \min_{\Lambda} \frac{|K|}{d(\Lambda)}, \quad (2)$$

the minimum is over all lattices  $\Lambda$  such that  $K + \Lambda$  is a  $j$ -fold lattice covering of  $\mathbb{R}^2$ .

A family  $\mathcal{F} = \{K_1, K_2, \dots\}$  of bounded sets which is both a  $j$ -fold packing and a  $j$ -fold covering of  $\mathbb{R}^2$  is called a *j-fold tiling* of  $\mathbb{R}^2$ . In addition, if each point of  $\mathbb{R}^2$  belongs to exactly  $j$  sets of the family, then we call  $\mathcal{F}$  an *exact j-fold tiling* of  $\mathbb{R}^2$ . For a fixed measurable bounded set  $K$ , we can define a *j-fold congruent tiling*, a *j-fold translative tiling*, a *j-fold lattice tiling*, an *exact j-fold congruent tiling*, an *exact j-fold translative tiling*, and an *exact j-fold lattice tiling* of  $\mathbb{R}^2$  with  $K$ . We call a bounded set  $K$  a *j-fold tile* if there exists a  $j$ -fold lattice tiling of  $\mathbb{R}^2$  with  $K$ , and call  $K$  an *exact j-fold tile* if there exists an exact  $j$ -fold lattice tiling of  $\mathbb{R}^2$  with  $K$ .

*Remark 1.1.* A 1-fold covering, a 1-fold packing and a 1-fold tiling are simply called a *covering*, a *packing* and a *tiling*, respectively.

It follows from the definitions that

$$j\delta_L(K) \leq \delta_L^j(K) \leq \delta_T^j(K) \leq \delta^j(K) \leq j \leq \vartheta^j(K) \leq \vartheta_T^j(K) \leq \vartheta_L^j(K) \leq j\vartheta_L(K)$$

where  $\delta_L(K) = \delta_L^1(K)$  and  $\vartheta_L(K) = \vartheta_L^1(K)$ . In addition, it is easy to see that  $\delta_T^j(K)$ ,  $\delta_L^j(K)$ ,  $\vartheta_T^j(K)$  and  $\vartheta_L^j(K)$  are invariant under non-singular affine transformations.

In 1972, Dumir and Hans-Gill [1][2] proved that both  $\delta_L^2(C) = 2\delta_L(C)$  and  $\vartheta_L^2(C) = 2\vartheta_L(C)$  hold for every centrally symmetric convex disk. Later, J. Pach introduced an idea to decompose complicated multiple packings and coverings to simpler ones. In 1984, G. Fejes Tóth [3] showed that every 3-fold lattice packing can be decomposed into three simple lattice packings and every 4-fold lattice packing can be decomposed into two 2-fold lattice packings. Furthermore, he obtained  $\delta_L^3(C) = 3\delta_L(C)$  and  $\delta_L^4(C) = 4\delta_L(C)$ .

As a special case, one can determine the  $j$ -fold lattice packing density and the  $j$ -fold lattice covering density of  $B^2$ , where  $B^2$  is the unit ball in  $\mathbb{R}^2$ , centered at the origin. The known results about  $\delta_L^j(B^2)$  and  $\vartheta_L^j(B^2)$  can be summarized in the following table [6].

$j$	$\delta_L^j(B^2)$	Author	$\vartheta_L^j(B^2)$	Author
1	$\frac{\pi}{\sqrt{12}}$	Lagrange	$\frac{2\pi}{\sqrt{27}}$	Kershner
2	$\frac{\pi}{\sqrt{3}}$	Heppes	$\frac{4\pi}{\sqrt{27}}$	Blundon
3	$\frac{3\pi}{2}$	Heppes	$\frac{\pi\sqrt{27138+2910\sqrt{97}}}{216}$	Blundon
4	$\frac{2\pi}{\sqrt{3}}$	Heppes	$\frac{25\pi}{18}$	Blundon
5	$\frac{2\pi}{\sqrt{7}}$	Blundon	$\frac{32}{7\sqrt{7}}$	Subak
6	$\frac{35\pi}{8\sqrt{6}}$	Blundon	$\frac{98}{27\sqrt{3}}$	Subak
7	$\frac{8\pi}{\sqrt{15}}$	Bolle	$7.672 \dots$	Haas
8	$\frac{3969\pi}{4\sqrt{(220-2\sqrt{193})(449+32\sqrt{193})}}$	Yakovlev	$\frac{32}{3\sqrt{15}}$	Temesvári
9	$\frac{25\pi}{2\sqrt{21}}$	Temesvári		

In this paper, I will determine the  $j$ -fold lattice packing density and the  $j$ -fold lattice covering density of a triangle  $T$ . The main results are as follows

**Theorem 1.2.** *For every triangle  $T$  and positive integer  $j$ ,*

$$\delta_L^j(T) = \frac{2j^2}{2j+1} \quad (3)$$

and

$$\vartheta_L^j(T) = \frac{2j+1}{2}. \quad (4)$$

Denote by  $\Delta_L^j(K)$  the collection of lattices  $\Lambda$  which  $K + \Lambda$  is a  $j$ -fold lattice packing of  $\mathbb{R}^2$  and the density of  $K + \Lambda$  is equal to  $\delta_L^j(K)$ . Denote by  $\Theta_L^j(K)$  the collection of lattices  $\Lambda$  which  $K + \Lambda$  is a  $j$ -fold lattice covering of  $\mathbb{R}^2$  and the density of  $K + \Lambda$  is equal to  $\vartheta_L^j(K)$ .

**Theorem 1.3.** *Suppose that  $T$  is the triangle of vertices  $(0, 0)$ ,  $(1, 0)$  and  $(0, 1)$ . We have that a lattice  $\Lambda$  is in  $\Delta_L^j(T)$  if and only if there exists an integer  $m$  such that  $1 \leq m \leq 2j + 1$ ,  $\gcd(m, 2j + 1) = \gcd(m + 1, 2j + 1) = 1$  and  $\Lambda$  is generated by  $\left(\frac{1}{2j}, \frac{m}{2j}\right)$  and  $\left(0, \frac{2j+1}{2j}\right)$ .*

**Theorem 1.4.** *Suppose that  $T$  is the triangle of vertices  $(0, 0)$ ,  $(1, 0)$  and  $(0, 1)$ . We have that a lattice  $\Lambda$  is in  $\Theta_L^j(T)$  if and only if there exists an integer  $m$  such that  $1 \leq m \leq 2j + 1$ ,  $\gcd(m, 2j + 1) = \gcd(m + 1, 2j + 1) = 1$  and  $\Lambda$  is generated by  $\left(\frac{1}{2j+1}, \frac{m}{2j+1}\right)$  and  $(0, 1)$ .*

**Corollary 1.5.** *For every triangle  $T$ , we have*

$$\text{card}\{\Delta_L^j(T)\} = \text{card}\{\Theta_L^j(T)\} = (2j + 1) \prod_{p|2j+1} \left(1 - \frac{2}{p}\right), \quad (5)$$

where the product is over the distinct prime numbers dividing  $2j + 1$ .

## 2 Some Definitions and Lemmas

From the definitions of  $j$ -fold lattice packing and covering, one can easily get the following lemma.

**Lemma 2.1.** *Let  $K$  be a convex disk,  $\Lambda$  be a lattice. We have*

1.  *$K + \Lambda$  is a  $j$ -fold lattice packing of  $\mathbb{R}^2$  if and only if for every point  $u$  in  $\mathbb{R}^2$ , there exist at most  $j$  distinct lattice points  $v_1, \dots, v_j$  in  $\Lambda$  such that  $u + v_1, \dots, u + v_j$  all belong to  $\text{Int}(K)$ .*
2.  *$K + \Lambda$  is a  $j$ -fold lattice covering of  $\mathbb{R}^2$  if and only if for every point  $u$  in  $\mathbb{R}^2$ , there exist at least  $j$  distinct lattice points  $v_1, \dots, v_j$  in  $\Lambda$  such that  $u + v_1, \dots, u + v_j$  all belong to  $K$ .*

**Definition 2.2.** Given a convex disk  $K$  and a lattice  $\Lambda$ , Let

$$\lambda^j(K, \Lambda) = \max\{l > 0 : lK + \Lambda \text{ is a } j\text{-fold lattice packing of } \mathbb{R}^2\}$$

and

$$\lambda_j(K, \Lambda) = \min\{l > 0 : lK + \Lambda \text{ is a } j\text{-fold lattice covering of } \mathbb{R}^2\}$$

In this section, we denote by  $T$  the triangle of vertices  $(0, 0)$ ,  $(1, 0)$  and  $(0, 1)$ . Let  $\Lambda$  be an arbitrary lattice and  $S_\Lambda$  is a fundamental domain of  $\Lambda$  (as shown in Figure 1). We note that  $S_\Lambda + \Lambda$  is an exact tiling of  $\mathbb{R}^2$ . Let

$$T^j(\Lambda) = \lambda^j(T, \Lambda) \cdot T,$$

and

$$T_j(\Lambda) = \lambda_j(T, \Lambda) \cdot T.$$

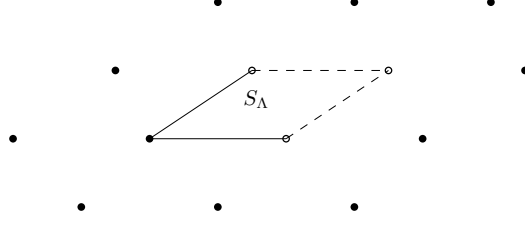


Figure 1:  $S_\Lambda$

For  $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ , we define the relation  $\prec$  by  $(x_1, y_1) \prec (x_2, y_2)$  if and only if either

$$x_1 + y_1 < x_2 + y_2$$

or

$$x_1 + y_1 = x_2 + y_2 \text{ and } x_1 < x_2.$$

*Remark 2.3.* For  $u, v, w \in \mathbb{R}^2$ , one can see that if  $u \neq v$ , then either  $u \prec v$  or  $v \prec u$ , and if  $u \prec v$ , then  $u + w \prec v + w$ .

Given a point  $u$  in  $\mathbb{R}^2$ , we define

$$V_j(u) = (u + \Lambda) \cap T_j(\Lambda).$$

Since  $T_j(\Lambda) + \Lambda$  is a  $j$ -fold lattice covering, by Lemma 2.1 we have  $\text{card}\{V_j(u)\} \geq j$ . We may assume, without loss of generality, that

$$V_j(u) = \{u_1, u_2, \dots, u_l\},$$

where  $l \geq j$  and  $u_1 \prec u_2 \prec \dots \prec u_l$ . Let

$$W_j(u) = \{u_1, u_2, \dots, u_j\},$$

and

$$S_j(\Lambda) = \bigcup_{u \in S_\Lambda} W_j(u),$$

where  $j = 1, 2, \dots$  and let  $S_0(\Lambda) = \emptyset$ .

**Lemma 2.4.** *Let  $u$  be a point in  $\mathbb{R}^2$  and  $v$  be a lattice point in  $\Lambda$ . Suppose that the  $x$ -coordinate and the  $y$ -coordinate of  $u + v$  both are non-negative. If  $u \in S_j(\Lambda)$  and  $u + v \prec u$ , then  $u + v \in S_j(\Lambda)$ .*

*Proof.* Since  $u \in S_j(\Lambda) \subset T_j(\Lambda)$  and  $u + v \prec u$ , we know that  $u + v \in T_j(\Lambda)$ . It follows from the definition of  $S_j(\Lambda)$  that  $u + v \in S_j(\Lambda)$ .  $\square$

**Lemma 2.5.** *Let  $u$  be a point in  $\mathbb{R}^2$ . Suppose that the  $x$ -coordinate and the  $y$ -coordinate of  $u$  both are non-negative. If  $u \notin S_j(\Lambda)$  then  $u' \prec u$  for all  $u' \in W_j(u)$ .*

*Proof.* From Lemma 2.4, we know that if  $u \prec u'$  for some  $u' \in W_j(u) \subset S_j(\Lambda)$ , then  $u \in S_j(\Lambda)$ .  $\square$

**Lemma 2.6.**  $Int(T^j(\Lambda)) \subset S_j(\Lambda) \subset T_j(\Lambda)$ .

*Proof.* Since  $W_j(u) \subset V_j(u) \subset T_j(\Lambda)$ , it is obvious that  $S_j(\Lambda) \subset T_j(\Lambda)$ . Now assume that there exists  $u \in Int(T^j(\Lambda)) \setminus S_j(\Lambda)$ . From Lemma 2.5, since  $u \notin S_j(\Lambda)$ , we have that for all  $u' \in W_j(u)$ ,  $u' \prec u$ . This implies that  $W_j(u) \subset Int(T^j(\Lambda))$ . We note that  $card\{W_j(u) \cup \{u\}\} = j+1$  and  $T^j(\Lambda) + \Lambda$  is a  $j$ -fold lattice packing of  $\mathbb{R}^2$ . From Lemma 2.1, one can see that this is impossible.  $\square$

**Lemma 2.7.**  $S_j(\Lambda) \subset S_{j+1}(\Lambda)$  and for every  $u \in \mathbb{R}^2$  there exists a unique  $v \in \Lambda$  such that  $u + v \in S_{j+1}(\Lambda) \setminus S_j(\Lambda)$ .

*Proof.* By the definition of  $W_j(u)$ , it is easy to see that  $W_j(u) \subset W_{j+1}(u)$  and  $card\{W_{j+1}(u) \setminus W_j(u)\} = 1$ . From the definition of  $S_j(\Lambda)$ , one can obtain the result.  $\square$

**Lemma 2.8.**  $(S_{j+1}(\Lambda) \setminus S_j(\Lambda)) + \Lambda$  is an exact tiling of  $\mathbb{R}^2$ .

*Proof.* This immediately follows from Lemma 2.7.  $\square$

**Lemma 2.9.**  $S_j(\Lambda) + \Lambda$  is an exact  $j$ -fold tiling of  $\mathbb{R}^2$ .

*Proof.* Note that  $S_j(\Lambda) = (S_j(\Lambda) \setminus S_{j-1}(\Lambda)) \cup (S_{j-1}(\Lambda) \setminus S_{j-2}(\Lambda)) \cup \dots \cup (S_2(\Lambda) \setminus S_1(\Lambda)) \cup S_1(\Lambda)$  and  $(S_i(\Lambda) \setminus S_{i-1}(\Lambda)) \cap (S_j(\Lambda) \setminus S_{j-1}(\Lambda)) = \emptyset$ , where  $i \neq j$ . Hence, the result immediately follows from Lemma 2.8.  $\square$

**Lemma 2.10.** Let  $u$  be a point in  $\mathbb{R}^2$  and  $v$  be a lattice point in  $\Lambda$ . Suppose that  $u \in S_{j+1}(\Lambda) \setminus S_j(\Lambda)$ . If  $u + v \in S_{j+1}(\Lambda)$  and  $v \neq (0, 0)$ , then  $u + v \in S_j(\Lambda)$ .

*Proof.* Assume that  $u + v \notin S_j(\Lambda)$ . Then  $u + v \in S_{j+1}(\Lambda) \setminus S_j(\Lambda)$ . But  $u \in S_{j+1}(\Lambda) \setminus S_j(\Lambda)$ , from Lemma 2.7, we know that  $u$  and  $u + v$  must be identical.  $\square$

**Lemma 2.11.** Let  $u$  be a point in  $\mathbb{R}^2$  and  $v$  be a lattice point in  $\Lambda$ . Suppose that  $u \in S_{j+1}(\Lambda) \setminus S_j(\Lambda)$ . If  $u \prec u + v$ , then  $u + v \notin S_{j+1}(\Lambda)$ .

*Proof.* Assume that  $u + v \in S_{j+1}(\Lambda)$ . From Lemma 2.10, since  $u \in S_{j+1}(\Lambda) \setminus S_j(\Lambda)$ , we know that  $u + v \in S_j(\Lambda)$ , i.e.,  $u + v \in W_j(u)$ . But  $u \notin S_j(\Lambda)$ , from Lemma 2.5, we have that  $u + v \prec u$ . This is a contradiction.  $\square$

**Lemma 2.12.** If  $(x, y) \in S_j(\Lambda)$ , then  $(x', y) \in S_j(\Lambda)$  and  $(x, y') \in S_j(\Lambda)$ , for all  $0 \leq x' \leq x$  and  $0 \leq y' \leq y$ .

*Proof.* Assume that  $x' < x$ . When  $j = 1$ , if  $(x', y) \notin S_1(\Lambda)$ , then there must be  $v \neq (0, 0)$  in  $\Lambda$  such that  $(x', y) + v \in S_1(\Lambda)$  and  $(x', y) + v \prec (x', y)$ . Hence  $(x, y) + v = (x', y) + v + (x - x', 0) \prec (x', y) + (x - x', 0) = (x, y)$ . This implies that  $(x, y) \notin S_1(\Lambda)$ . This is a contradiction, and hence  $(x', y) \in S_1(\Lambda)$ , for all  $0 \leq x' \leq x$ . By the similar reason, we have  $(x, y') \in S_1(\Lambda)$ , for all  $0 \leq y' \leq y$ .

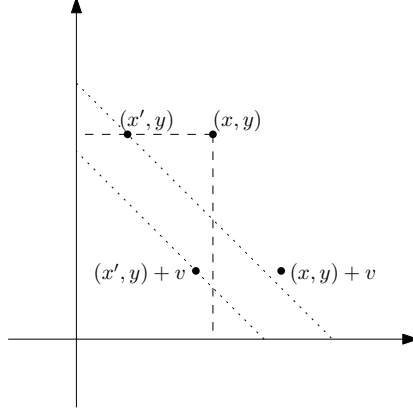


Figure 2:  $(x, y)$  and  $(x, y) + v$

Now we assume that the lemma is true for  $j = k$ . We may suppose that  $(x, y) \in S_{k+1}(\Lambda) \setminus S_k(\Lambda)$ . If there exists  $0 \leq x' < x$  such that  $(x', y) \notin S_{k+1}(\Lambda)$ , then by Lemma 2.8 we have that there must be  $v \neq (0, 0)$  in  $\Lambda$  such that  $(x', y) + v \in S_{k+1}(\Lambda) \setminus S_k(\Lambda)$  and hence  $(x', y) + v \prec (x', y)$ . Therefore,  $(x, y) + v \prec (x, y)$ . Since  $(x, y) \in S_{k+1}(\Lambda) \setminus S_k(\Lambda)$ , from Lemma 2.4 and Lemma 2.10, we know that  $(x, y) + v \in S_k(\Lambda)$ . By the inductive hypothesis, we have that  $(x', y) + v \in S_k(\Lambda)$ . This is a contradiction.  $\square$

We call a set  $S$  a *half open  $r$ -stair polygon* if there are  $x_0 < x_1 < \dots < x_{r+1}$  and  $y_0 > y_1 > \dots > y_r > y_{r+1}$  such that

$$S = \bigcup_{i=0}^r [x_i, x_{i+1}) \times [y_{r+1}, y_i)$$

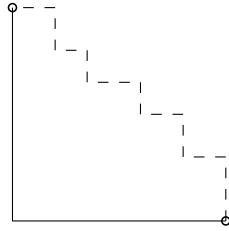


Figure 3: a half open 4-stair polygon

**Lemma 2.13.**  $S_j(\Lambda)$  is a half open stair polygon.

*Proof.* From Lemma 2.12, we know that  $S_1(\Lambda)$  must be in the shape as shown in Figure 4. Furthermore, by Lemma 2.8, we have that  $S_1(\Lambda)$  is an exact tile.

Hence, it is not hard to see that  $S_1(\Lambda)$  must be a half open stair polygon. Note that  $S_j(\Lambda) \subset S_{j+1}(\Lambda)$ , by using mathematical induction on  $j$  and Lemma 2.8, one can easily obtain the result.  $\square$

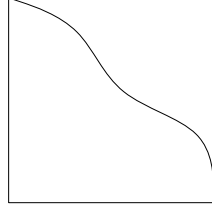


Figure 4: a possible shape of  $S_1(\Lambda)$

From Lemma 2.13, we may assume that

$$S_j(\Lambda) = \bigcup_{i=0}^{r_j} [x_i^{(j)}, x_{i+1}^{(j)}) \times [0, y_i^{(j)}),$$

where  $0 = x_0^{(j)} < x_1^{(j)} < \dots < x_{r_i+1}^{(j)}$  and  $y_0^{(j)} > y_1^{(j)} > \dots > y_{r_i}^{(j)} > 0$ . Let

$$Z_j^*(\Lambda) = \{(0, y_0^{(j)}), (x_{r+1}^{(j)}, 0)\},$$

and

$$Z_j(\Lambda) = \{(x_i^{(j)}, y_i^{(j)}) : i = 1, \dots, r_j\}.$$

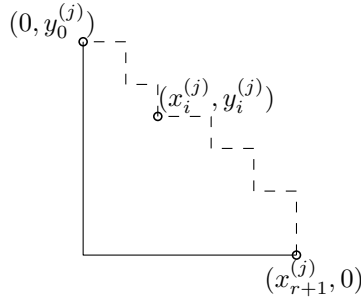


Figure 5:  $S_j(\Lambda)$

**Lemma 2.14.** *Let  $v \neq (0, 0)$  be a lattice point in  $\Lambda$  and  $u$  is a point in  $\mathbb{R}^2$ . If  $u$  and  $u + v$  both are in  $Z_{j+1}(\Lambda)$ , then  $u \in Z_j(\Lambda)$  or  $u + v \in Z_j(\Lambda)$ .*

*Proof.* Assume that  $u \notin Z_j(\Lambda)$ . Without loss of generality, we may assume that there exists  $\varepsilon > 0$  such that  $u - (0, \varepsilon') \in S_{j+1}(\Lambda) \setminus S_j(\Lambda)$ , for all  $0 < \varepsilon' < \varepsilon$ . Since  $u + v \in Z_{j+1}(\Lambda)$ , there must exist  $0 < \varepsilon_0 < \varepsilon$  such that  $u + v - (0, \varepsilon_0) \in$



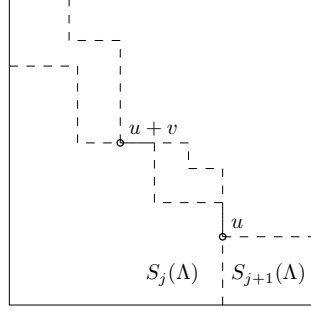


Figure 6:  $S_j(\Lambda)$  and  $S_{j+1}(\Lambda)$

$S_{j+1}(\Lambda)$ . From Lemma 2.10, since  $u - (0, \varepsilon_0) \in S_{j+1}(\Lambda) \setminus S_j(\Lambda)$ , we have that  $u + v - (0, \varepsilon_0) \in S_j(\Lambda)$ . These can be deduced that  $u + v \prec u$ . Similarly, if  $u + v \notin Z_j(\Lambda)$ , then  $u \prec u + v$ . It is obvious that  $u \prec u + v$  and  $u + v \prec u$  cannot occur simultaneously. Hence, we have that  $u \in Z_j(\Lambda)$  or  $u + v \in Z_j(\Lambda)$ .  $\square$

**Lemma 2.15.** *For every  $u \in Z_{j+1}(\Lambda) \setminus Z_j(\Lambda)$ , there exists a unique lattice point  $v \neq (0, 0)$  in  $\Lambda$  such that  $u + v \in (Z_j^*(\Lambda) \cup Z_j(\Lambda)) \setminus Z_{j+1}(\Lambda)$ .*

*Proof.* It is clear that  $u \notin S_{j+1}(\Lambda)$ , and hence there is a unique lattice point  $v \neq (0, 0)$  in  $\Lambda$  such that  $u + v \in S_{j+1}(\Lambda) \setminus S_j(\Lambda)$  and  $u + v \prec u$ . Obviously,  $u + v \notin Z_{j+1}(\Lambda)$ . If  $u + v \notin Z_j^*(\Lambda) \cup Z_j(\Lambda)$ , then we may assume, without loss of generality, that there exists  $\varepsilon > 0$  such that  $u + v - (0, \varepsilon') \in S_{j+1}(\Lambda) \setminus S_j(\Lambda)$ , for all  $0 < \varepsilon' < \varepsilon$ . Since  $u + v \prec u$ , we know that for every  $0 < \varepsilon' < \varepsilon$ ,  $u + v - (0, \varepsilon') \prec u - (0, \varepsilon')$ . From Lemma 2.11, we have that  $u - (0, \varepsilon') \notin S_{j+1}(\Lambda)$  for every  $0 < \varepsilon' < \varepsilon$ . This is impossible, since  $u \in Z_{j+1}(\Lambda)$ . Hence  $u + v \in (Z_j^*(\Lambda) \cup Z_j(\Lambda)) \setminus Z_{j+1}(\Lambda)$ .  $\square$

**Lemma 2.16.**  $\text{card}\{Z_j(\Lambda)\} \leq 2j - 1$ .

*Proof.* When  $j = 1$ , since  $S_1(\Lambda) + \Lambda$  is a tiling of  $\mathbb{R}^2$ , it is easy to show that  $\text{card}\{(Z_1(\Lambda))\} \leq 1$ . Now assume that  $\text{card}\{Z_k(\Lambda)\} \leq 2k - 1$ . From Lemma 2.14 and Lemma 2.15, one can deduce that

$$\text{card}\{Z_{k+1}(\Lambda) \setminus Z_k(\Lambda)\} \leq \text{card}\{(Z_k^*(\Lambda) \cup Z_k(\Lambda)) \setminus Z_{k+1}(\Lambda)\}.$$

We note that

$$\text{card}\{Z_{k+1}(\Lambda)\} = \text{card}\{Z_{k+1}(\Lambda) \setminus Z_k(\Lambda)\} + \text{card}\{Z_{k+1}(\Lambda) \cap Z_k(\Lambda)\},$$

and

$$\begin{aligned} \text{card}\{Z_k^*(\Lambda) \cup Z_k(\Lambda)\} &= \text{card}\{(Z_k^*(\Lambda) \cup Z_k(\Lambda)) \setminus Z_{k+1}(\Lambda)\} \\ &\quad + \text{card}\{Z_{k+1}(\Lambda) \cap Z_k(\Lambda)\}. \end{aligned}$$

Hence

$$\text{card}\{Z_{k+1}(\Lambda)\} \leq \text{card}\{Z_k^*(\Lambda) \cup Z_k(\Lambda)\} \leq 2 + 2k - 1 = 2(k+1) - 1.$$

□

Denote by  $\mathcal{S}_j$  the collection of half open  $r$ -stair polygons  $S$  contained in  $T$  which  $r \leq 2j - 1$  and  $S$  is an exact  $j$ -fold tile. Denote by  $\mathcal{S}^j$  the collection of half open  $r$ -stair polygons  $S$  such that  $\text{Int}(T) \subset S$ ,  $r \leq 2j - 1$  and  $S$  is an exact  $j$ -fold tile. Let  $A_j$  denote the maximum area of polygons in  $\mathcal{S}_j$  and let  $A^j$  denote the minimum area of polygons in  $\mathcal{S}^j$ .

For any given  $S \in \mathcal{S}_j$ , suppose that  $S + \Lambda$  is a  $j$ -fold lattice tiling of  $\mathbb{R}^2$ . Since  $S \subset T$ , it is easy to see that  $T + \Lambda$  is a  $j$ -fold lattice covering of  $\mathbb{R}^2$ . Clearly, the density of  $T + \Lambda$  is  $\frac{|T|}{d(\Lambda)} = \frac{j|T|}{|S|}$ . Hence

$$\vartheta_L^j(T) \leq \frac{j|T|}{|S|},$$

for all  $S \in \mathcal{S}_j$ . Therefore,

$$\vartheta_L^j(T) \leq \frac{j|T|}{A_j}.$$

Similarly, one can show that

$$\delta_L^j(T) \geq \frac{j|T|}{A^j}.$$

For any given lattice  $\Lambda$ , by the definition of  $\lambda_j(T, \Lambda)$ , Lemma 2.6, Lemma 2.9, Lemma 2.13 and Lemma 2.16, we know that  $\frac{1}{\lambda_j(T, \Lambda)} S_j(\Lambda) \in \mathcal{S}_j$ . From (2), we can obtain

$$\vartheta_L^j(T) = \min_{\Lambda} \frac{|T_j(\Lambda)|}{d(\Lambda)} = \min_{\Lambda} \frac{j|T_j(\Lambda)|}{|S_j(\Lambda)|} = \min_{\Lambda} \frac{j|T|}{\left| \frac{1}{\lambda_j(T, \Lambda)} S_j(\Lambda) \right|} \geq \min_{S \in \mathcal{S}_j} \frac{j|T|}{|S|} = \frac{j|T|}{A_j}$$

here, the minima are over all lattices  $\Lambda$ . Hence

$$\vartheta_L^j(T) = \frac{j|T|}{A_j}. \quad (6)$$

Similarly, one can show that

$$\delta_L^j(T) = \frac{j|T|}{A^j}. \quad (7)$$

### 3 $j$ -Fold Tiling with Stair Polygon

Let  $S(j)$  be a half open  $(2j - 1)$ -stair polygon defined by

$$S(j) = \bigcup_{i=0}^{2j-1} [i, i+1) \times [0, 2j-i).$$

In this section, we will prove the following result.

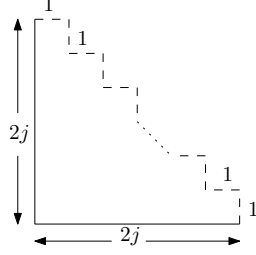


Figure 7:  $S(j)$

**Theorem 3.1.**  $S(j)$  is an exact  $j$ -fold tile. Furthermore,  $S(j) + \Lambda$  is an exact  $j$ -fold lattice tiling of  $\mathbb{R}^2$  if and only if there exists an integer  $1 \leq m \leq 2j+1$  such that  $\gcd(m, 2j+1) = \gcd(m+1, 2j+1) = 1$  and  $\Lambda$  is generated by  $(1, m)$  and  $(0, 2j+1)$ .

Let

$$S^*(j) = \bigcup_{i=1}^{2j} [i, i+1) \times [2j+1-i, 2j+1),$$

and

$$D(j) = \bigcup_{i=0}^{2j} [i, i+1) \times [2j-i, 2j+1-i).$$

Denote by  $U(j)$  the set  $[0, 2j+1) \times [0, 2j+1)$ . Clearly,  $S(j)$ ,  $S^*(j)$  and  $D(j)$  are mutually disjoint, and

$$U(j) = S(j) \cup D(j) \cup S^*(j).$$

Let

$$B(j) = [0, 1) \times [0, 2j+1),$$

and

$$C(j) = [0, 2j+1) \times [0, 1).$$

For any given lattice  $\Lambda$ , from the definition, one can see that  $S(j) + \Lambda$  is an exact  $j$ -fold tiling of  $\mathbb{R}^2$  if and only if for every point  $(x, y)$  in  $\mathbb{R}^2$ ,  $\text{card}\{((x, y) + \Lambda) \cap S(j)\} = j$ , i.e.,  $\text{card}\{\Lambda \cap (-(x, y) + S(j))\} = j$ . This can be interpreted as  $\text{card}\{\Lambda \cap \tau(S(j))\} = j$ , for all translations  $\tau$ . Let  $m$  be a positive integer. Denote by  $\Lambda(m, j)$  the lattice generated by  $(1, m)$  and  $(0, 2j+1)$ .

**Lemma 3.2.**  $\text{card}\{\Lambda(m, j) \cap ((s, t) + B(j))\} = 1$ , for all  $(s, t) \in \mathbb{Z}^2$ .

*Proof.* For any given  $(s, t) \in \mathbb{Z}^2$ , we determine the equation

$$\begin{cases} c_1 \cdot 1 + c_2 \cdot 0 = s \\ c_1 \cdot m + c_2 \cdot (2j+1) = l + t \end{cases} \quad (8)$$

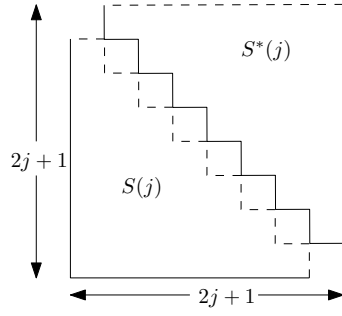


Figure 8:  $S(j)$  and  $S^*(j)$

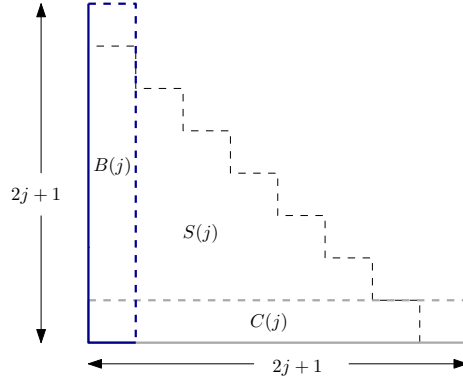


Figure 9:  $B(j)$  and  $C(j)$

where  $l = 0, 1, \dots, 2j$ . One can obtain

$$c_1 = s,$$

and

$$c_2 = \frac{l + t - sm}{2j + 1}.$$

By elementary number theory, there exists a unique  $l \in \{0, 1, \dots, 2j\}$  such that  $\frac{l+t-sm}{2j+1}$  is an integer.  $\square$

**Lemma 3.3.**  $\text{card}\{\Lambda(m, j) \cap ((s, t) + C(j))\} = \gcd(m, 2j+1)$ , for all  $(s, t) \in \mathbb{Z}^2$ .

*Proof.* Let  $d = \gcd(m, 2j+1)$ . For any given  $(s, t) \in \mathbb{Z}^2$ , we determine the equation

$$\begin{cases} c_1 \cdot 1 + c_2 \cdot 0 = l + s \\ c_1 \cdot m + c_2 \cdot (2j + 1) = t \end{cases} \quad (9)$$

where  $l = 0, 1, \dots, 2j$ . One can obtain

$$c_1 = l + s,$$

and

$$c_2 = \frac{t - ms - ml}{2j + 1}.$$

By elementary number theory, there exist exactly  $d$  numbers of  $l \in \{0, 1, \dots, 2j\}$  such that  $\frac{t - ms - ml}{2j + 1}$  is an integer.  $\square$

**Lemma 3.4.**  $\text{card}\{\Lambda(m, j) \cap ((s, t) + D(j))\} = \gcd(m + 1, 2j + 1)$ , for all  $(s, t) \in \mathbb{Z}^2$ .

*Proof.* Let  $d = \gcd(m + 1, 2j + 1)$ . Determine the equation

$$\begin{cases} c_1 \cdot 1 + c_2 \cdot 0 = l + s \\ c_1 \cdot m + c_2 \cdot (2j + 1) = 2j - l + t \end{cases} \quad (10)$$

where  $l = 0, 1, \dots, 2j$  and  $(s, t) \in \mathbb{Z}^2$ . One can get

$$c_1 = l + s,$$

and

$$c_2 = \frac{2j - sm + t - (m + 1)l}{2j + 1}.$$

By elementary number theory, we know that there are exactly  $d$  numbers of  $l$  in  $\{0, 1, \dots, 2j\}$  such that  $\frac{2j - sm + t - (m + 1)l}{2j + 1}$  is an integer. Hence there are exactly  $d$  lattice points in  $\Lambda(m, j) \cap ((s, t) + D(j))$ .  $\square$

**Lemma 3.5.** Suppose that  $m$  satisfies  $1 \leq m \leq 2j + 1$  and  $\gcd(m, 2j + 1) = \gcd(m + 1, 2j + 1) = 1$ . Given a  $(s, t) \in \mathbb{Z}^2$ . If

$$\text{card}\{\Lambda(m, j) \cap ((s, t) + S(j))\} = k,$$

then

$$\text{card}\{\Lambda(m, j) \cap ((s', t') + S(j))\} = k,$$

for every  $(s', t') \in \mathbb{Z}^2$ .

*Proof.* It suffices to show that  $\text{card}\{\Lambda(m, j) \cap ((s, t) + u + S(j))\} = k$ , where  $u = (0, 1), (1, 0)$ . Suppose that  $u = (0, 1)$ . One can see that

$$(s, t + 1) + S(j) = (s, t) + ((S(j) \cup D(j)) \setminus C(j)).$$

Since  $\gcd(m, 2j + 1) = \gcd(m + 1, 2j + 1) = 1$ , by Lemma 3.3 and Lemma 3.4, we know that

$$\text{card}\{\Lambda(m, j) \cap ((s, t) + C(j))\} = 1,$$

and

$$\text{card}\{\Lambda(m, j) \cap ((s, t) + D(j))\} = 1.$$

Clearly,  $S(j) \cap C(j) \cap D(j) = \emptyset$ . From these, one can deduce that

$$\text{card}\{\Lambda(m, j) \cap ((s, t + 1) + S(j))\} = \text{card}\{\Lambda(m, j) \cap ((s, t) + S(j))\} = k$$

When  $u = (1, 0)$ . We have

$$(s+1, t) + S(j) = (s, t) + ((S(j) \cup D(j)) \setminus B(j)).$$

By using Lemma 3.2, we can obtain

$$\text{card}\{\Lambda(m, j) \cap ((s+1, t) + S(j))\} = \text{card}\{\Lambda(m, j) \cap ((s, t) + S(j))\} = k$$

□

**Lemma 3.6.** Suppose that  $m$  satisfies  $1 \leq m \leq 2j+1$ ,  $\gcd(m, 2j+1) = 1$  and  $\gcd(m+1, 2j+1) = d$ . We have that

$$\text{card}\{\Lambda(m, j) \cap S(j)\} = j+1-d.$$

*Proof.* We note that

$$\Lambda(m, j) \cap U(j) = \bigcup_{k=0}^{2j} (\Lambda(m, j) \cap (u_k + B(j))).$$

where  $u_k$  denotes the point  $(k, 0)$ . From Lemma 3.2, one can see that

$$\text{card}\{\Lambda(m, j) \cap U(j)\} = 2j+1.$$

Obviously,  $(0, 0) \in \Lambda(m, j) \cap S(j)$ . Since  $\gcd(m, 2j+1) = 1$ , it is not hard to prove that for  $1 \leq k \leq 2j$ ,  $(k, 2j+1-k)$  cannot be in  $\Lambda(m, j)$ . Furthermore, one can show that when  $1 \leq k \leq 2j$ , if  $\Lambda(m, j) \cap D(j) \cap (u_k + B(j)) = \emptyset$  and  $\Lambda(m, j) \cap D(j) \cap (u_{2j+1-k} + B(j)) = \emptyset$ , then we have  $\text{card}\{\Lambda(m, j) \cap S(j) \cap (u_k + B(j))\} + \text{card}\{\Lambda(m, j) \cap S(j) \cap (u_{2j+1-k} + B(j))\} = 1$  (see Figure 10). If  $\Lambda(m, j) \cap D(j) \cap (u_k + B(j)) \neq \emptyset$ , then  $\text{card}\{\Lambda(m, j) \cap S(j) \cap (u_k + B(j))\} = \text{card}\{\Lambda(m, j) \cap S(j) \cap (u_{2j+1-k} + B(j))\} = 0$  (see Figure 11, we note that  $\text{card}\{\Lambda(m, j) \cap S^*(j) \cap (u_{2j+1-k} + B(j))\} = 1$ ).

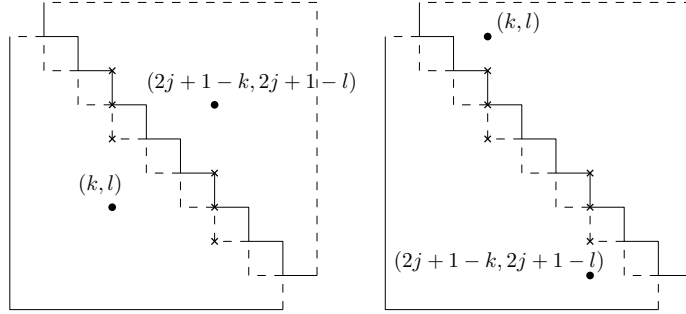


Figure 10: The case  $\Lambda(m, j) \cap D(j) \cap (u_k + B(j)) = \emptyset$  and  $\Lambda(m, j) \cap D(j) \cap (u_{2j+1-k} + B(j)) = \emptyset$ , where  $(k, l), (2j+1-k, 2j+1-l) \in \Lambda(m, j)$

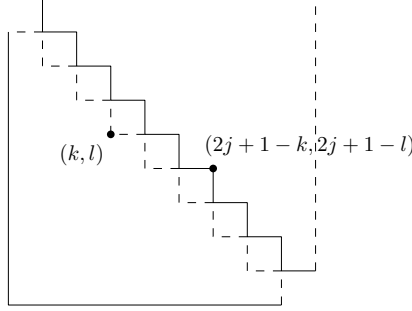


Figure 11: The case  $\Lambda(m, j) \cap D(j) \cap (u_k + B(j)) \neq \emptyset$ , where  $(k, l), (2j+1-k, 2j+1-l) \in \Lambda(m, j)$

By Lemma 3.4, we know that there exist exactly  $d$  numbers of  $k_1, \dots, k_d \in \{1, 2, \dots, 2j\}$  such that  $\Lambda(m, j) \cap D(j) \cap (u_{k_i} + B(j)) \neq \emptyset$ , for  $i = 1, 2, \dots, d$ . Therefore, we have  $\text{card}\{\Lambda(m, j) \cap S(j) \cap (u_{k_i} + B(j))\} = \text{card}\{\Lambda(m, j) \cap S(j) \cap (u_{2j+1-k_i} + B(j))\} = 0$ , for  $i = 1, 2, \dots, d$ . Furthermore, when  $k \in \{1, 2, \dots, 2j\} \setminus \{k_1, \dots, k_d, 2j+1-k_1, \dots, 2j+1-k_d\}$ , we have  $\text{card}\{\Lambda(m, j) \cap S(j) \cap (u_k + B(j))\} + \text{card}\{\Lambda(m, j) \cap S(j) \cap (u_{2j+1-k} + B(j))\} = 1$ . This can be deduced that

$$\text{card}\{\Lambda(m, j) \cap (S(j) \setminus B(j))\} = \frac{2j-2d}{2} = j-d,$$

and hence

$$\text{card}\{\Lambda(m, j) \cap S(j)\} = j+1-d.$$

□

**Lemma 3.7.** Suppose that  $m$  satisfies  $1 \leq m \leq 2j+1$  and  $\gcd(m, 2j+1) = \gcd(m+1, 2j+1) = 1$ . For every  $(s, t) \in \mathbb{Z}^2$ , we have

$$\text{card}\{\Lambda(m, j) \cap ((s, t) + S(j))\} = j,$$

*Proof.* From Lemma 3.6, we have that

$$\text{card}\{\Lambda(m, j) \cap S(j)\} = j.$$

Hence, it immediately follows from Lemma 3.5 that

$$\text{card}\{\Lambda(m, j) \cap ((s, t) + S(j))\} = j,$$

for every  $(s, t) \in \mathbb{Z}^2$ .

□

**Lemma 3.8.** Suppose that  $m$  satisfies  $1 \leq m \leq 2j+1$  and  $\gcd(m, 2j+1) = \gcd(m+1, 2j+1) = 1$ . Then for every  $(x, y) \in \mathbb{R}^2$ ,

$$\text{card}\{\Lambda(m, j) \cap ((x, y) + S(j))\} = j,$$

*Proof.* Suppose that  $s - 1 < x \leq s$  and  $t - 1 < y \leq t$ , where  $s, t \in \mathbb{Z}$ . One can observe that

$$\Lambda(m, j) \cap ((x, y) + S(j)) = \Lambda(m, j) \cap ((s, t) + S(j))$$

From Lemma 3.7, we obtain

$$\text{card}\{\Lambda(m, j) \cap ((x, y) + S(j))\} = j.$$

□

It immediately follows from Lemma 3.8 that  $S(j) + \Lambda(m, j)$  is an exact  $j$ -fold lattice tiling of  $\mathbb{R}^2$ , when  $1 \leq m \leq 2j+1$  and  $\gcd(m, 2j+1) = \gcd(m+1, 2j+1) = 1$ . In order to complete the proof of Theorem 3.1, we will prove the following lemmas.

**Lemma 3.9.** *If  $S(j) + \Lambda$  is an exact  $j$ -fold lattice tiling of  $\mathbb{R}^2$ , then there exist real numbers  $-1 < x \leq 2j - 1$  and  $-1 < y \leq 2j - 1$  such that both  $(x, 1)$  and  $(1, y)$  are in  $\Lambda$ .*

*Proof.* Let  $u = (1, 2j - 1)$ . Denote by  $V$  the collection of lattice points  $v$  in  $\Lambda$  such that  $u \in S(j) + v$ . Since  $S(j) + \Lambda$  is an exact  $j$ -fold lattice tiling, we know that  $\text{card}\{V\} = j$ . Let

$$s_0 = \max\{s : (s, t) \in V\}$$

Obviously,  $s_0 \leq 1$ . If  $s_0 < 1$ , then choose  $0 < \varepsilon < \min\{1, 1 - s_0\}$ . It is easy to see that  $u - (\varepsilon, 0) \in S(j) + v$ , for all  $v \in V$ . Furthermore, it is obvious that  $u - (\varepsilon, 0) \in S(j)$ , but  $(0, 0) \notin V$ . This implies that  $\text{card}\{((u - (\varepsilon, 0)) + \Lambda) \cap S(j)\} \geq j + 1$ . This is a contradiction. Hence,  $s_0 = 1$ , i.e., there exists a real number  $y$  such that  $(1, y) \in V$ . It is easy to see that  $-1 < y \leq 2j - 1$ . By determining the point  $(2j - 1, 1)$ , one can show that  $(x, 1) \in \Lambda$ , for some  $-1 < x \leq 2j - 1$ . □

**Lemma 3.10.** *Suppose that  $S(j) + \Lambda$  is an exact  $j$ -fold lattice tiling of  $\mathbb{R}^2$  and  $s > 0$ . If  $(s, 0) \in \Lambda$  or  $(0, s) \in \Lambda$ , then  $s \geq 2j + 1$ .*

*Proof.* Since  $S(j) + \Lambda$  is an exact  $j$ -fold lattice tiling of  $\mathbb{R}^2$ , one can see that  $\frac{|S(j)|}{d(\Lambda)} = j$ . Hence  $d(\Lambda) = 2j + 1$ . Without loss of generality, we assume that  $(s, 0) \in \Lambda$ . By Lemma 3.9, there exists  $x$  such that  $(x, 1) \in \Lambda$ . By the property of  $d(\Lambda)$ , it is clear that

$$s = \begin{vmatrix} s & x \\ 0 & 1 \end{vmatrix}$$

must be greater than or equal to  $2j + 1$ . □

**Lemma 3.11.** *If  $S(j) + \Lambda$  is an exact  $j$ -fold lattice tiling of  $\mathbb{R}^2$ , then there exist  $s, t \in \{1, 2, \dots, 2j\}$  such that  $(-s, s - 1)$  and  $(t - 1, -t)$  both are in  $\Lambda$ .*



*Proof.* Let  $u = (0, 2j)$ . Denote by  $V(u)$  the collection of lattice points  $v$  in  $\Lambda$  such that  $u \in S(j) + v$ . Then  $\text{card}\{V(u)\} = j$ . Let

$$b_0 = \max\{b : (a, b) \in V(u)\}$$

Obviously,  $b_0 \leq 2j$ . If  $b_0 < 2j$ , then choose  $0 < \varepsilon < \min\{1, 2j - b_0\}$ . It is easy to see that  $u - (0, \varepsilon) \in S(j) + v$ , for all  $v \in V(u)$ . Note that  $(0, 0) \notin V(u)$ , but  $u - (0, \varepsilon) \in S(j)$ . This is a contradiction, since  $S(j) + \Lambda$  is an exact  $j$ -fold lattice tiling of  $\mathbb{R}^2$ . Hence  $b_0 = 2j$ . From Lemma 3.10, it follows that there is exactly one  $(a, b) \in V(u)$  such that  $b = 2j$ . Assume that  $(a_0, 2j) \in V(u)$ . Clearly,  $-2j < a_0 \leq 0$ . Again, by Lemma 3.10, it is not hard to see that  $a_0 \neq 0$  and  $u \in \text{Int}(S(j) + v)$ , whenever  $v \in V(u) \setminus \{(a_0, 2j)\}$ . Therefore, there exists  $0 < \varepsilon_0 < -a_0$  such that for all  $0 < \varepsilon' \leq \varepsilon_0$  and  $v \in V(u) \setminus \{(a_0, 2j)\}$ ,  $u - (0, \varepsilon') \in S(j) + v$  and  $u - (\varepsilon', 0) \in S(j) + v$ . This can be deduced that for every  $0 < \varepsilon' \leq \varepsilon_0$ ,  $V(u - (0, \varepsilon')) = (V(u) \cup \{(0, 0)\}) \setminus \{(a_0, 2j)\}$  and  $V(u - (\varepsilon', 0)) = V(u)$ .

So until now, if let

$$\mathcal{F} = \{S(j) + v : v \in V(u) \cup \{(0, 0)\}\},$$

then we have that

- (i) there are exactly  $j$  polygons in  $\mathcal{F}$  that contain the line segment  $\{u - (\varepsilon', 0) : 0 < \varepsilon' < \varepsilon_0\}$ .
- (ii) there are exactly  $j$  polygons in  $\mathcal{F}$  that contain the line segment  $\{u - (0, \varepsilon') : 0 < \varepsilon' < \varepsilon_0\}$ .
- (iii) there are exactly  $j - 1$  polygons in  $\mathcal{F}$  that contain the square  $\tilde{U} = \{u - (\varepsilon'_1, \varepsilon'_2) : 0 < \varepsilon'_i < \varepsilon_0, i = 1, 2\}$  ( Here, we note that  $S(j) \cap \tilde{U} = \emptyset$  and  $(S(j) + (a_0, 2j)) \cap \tilde{U} = \emptyset$ ).

From these, one can see that there must exist an integer  $1 \leq s \leq 2j$  and a lattice point  $v \in \Lambda$  such that  $(s, 2j + 1 - s) + v = u$ , i.e.,  $v = (-s, s - 1)$  (see Figure 12). By symmetry, one can obtain that  $(t - 1, -t) \in \Lambda$ , for some  $t \in \{1, 2, \dots, 2j\}$ .  $\square$

Now we will prove the remaining part of Theorem 3.1. Suppose that  $S(j) + \Lambda$  is an exact  $j$ -fold lattice tiling of  $\mathbb{R}^2$ . From Lemma 3.11, we may assume that  $(-s, s - 1), (t - 1, -t) \in \Lambda$ , for some  $s, t \in \{1, 2, \dots, 2j\}$ . We note that  $d(\Lambda) = 2j + 1$ . Hence

$$s + t - 1 = \begin{vmatrix} -s & t - 1 \\ s - 1 & -t \end{vmatrix}$$

must be divisible by  $2j + 1$ . Since  $0 < s + t - 1 \leq 4j - 1$ , we have  $s + t - 1 = 2j + 1$ , i.e.,  $\Lambda$  can be generated by  $(-s, s - 1)$  and  $(t - 1, -t)$ . From Lemma 3.9, there exists a real number  $-1 < m \leq 2j - 1$  such that  $(1, m) \in \Lambda$ . Since  $s$  and  $t$  both are integers, we have that  $m$  is also an integer. By determining the equation

$$\begin{cases} c_1 \cdot (-s) + c_2 \cdot (t - 1) = 0 \\ c_1 \cdot (s - 1) + c_2 \cdot (-t) = 2j + 1 \end{cases} \quad (11)$$

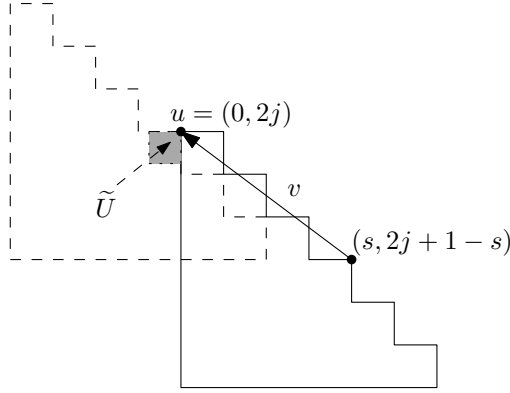


Figure 12:  $(s, 2j + 1 - s) + v = u$

One can see that  $(0, 2j + 1) \in \Lambda$ , and hence  $\Lambda$  can be generated by  $(1, m)$  and  $(0, 2j + 1)$ , where  $m \in \{0, 1, \dots, 2j - 1\}$ . From Lemma 3.10, we know that  $m \neq 0$ , i.e.,  $m \in \{1, 2, \dots, 2j - 1\}$ . Again, from Lemma 3.9 and Lemma 3.10, since  $s, t$  are integers, there must exist an integer  $1 \leq n \leq 2j - 1$  such that  $(n, 1) \in \Lambda$ . If  $\gcd(m, 2j + 1) \neq 1$ , then we can choose an integer  $k$  satisfies  $0 \leq k \leq 2j$  and  $mn - k$  is divisible by  $2j + 1$ . One can see that  $(n, k) \in \Lambda$  and  $k \neq 1$ . Hence  $(n, 1), (n, k) \in \Lambda \cap ((n, 0) + B(j))$ . From Lemma 3.2, we know that this is impossible. Therefore,  $\gcd(m, 2j + 1) = 1$ . Now we suppose that  $\gcd(m + 1, 2j + 1) = d$ . From Lemma 3.6, we know that

$$\text{card}\{\Lambda \cap S(j)\} = j + 1 - d.$$

Since  $S(j) + \Lambda$  is an exact  $j$ -fold lattice tiling of  $\mathbb{R}^2$ , we have

$$\text{card}\{\Lambda \cap S(j)\} = j.$$

Hence  $\gcd(m + 1, 2j + 1) = d = 1$ . This completes the proof of Theorem 3.1.

## 4 Generalized Euler $\varphi$ Function

**Definition 4.1.** An *arithmetic function* is a function that is defined for all positive integers.

**Definition 4.2.** An arithmetic function  $f$  is called *multiplicative* if  $f(mn) = f(m)f(n)$  whenever  $m$  and  $n$  are relatively prime positive integers.

We have the following elementary result.

**Theorem 4.3.** If  $f$  is a multiplicative function and if  $n = p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s}$  is the prime power factorization of the positive integer  $n$ , then

$$f(n) = f(p_1^{a_1}) f(p_2^{a_2}) \cdots f(p_s^{a_s}).$$

To find the number of  $m$  that satisfies the conditions in Theorem 3.1, we determine the following arithmetic function

$$\varphi^k(n) = \text{card}\{m : 1 \leq m \leq n, \gcd(m, n) = \cdots = \gcd(m + k - 1, n) = 1\}$$

When  $k = 1$ ,  $\varphi^1$  is the well-known Euler Phi function. It is not hard to prove that  $\varphi^k$  is a multiplicative function. Furthermore, when  $p$  is a prime number and  $a$  is a positive integer, if  $p > k$  then  $\varphi^k(p^a) = p^{a-1}(p - k)$ , and if  $p \leq k$  then  $\varphi^k(p^a) = 0$ . From Theorem 4.3, we can obtain the following result.

**Theorem 4.4.** *Let  $n = p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s}$  be the prime power factorization of the positive integer  $n$ . Then*

$$\varphi^k(n) = \begin{cases} n(1 - \frac{k}{p_1}) \cdots (1 - \frac{k}{p_s}) & p_i > k \text{ for all } i = 1, \dots, s, \\ 0 & p_i \leq k \text{ for some } i. \end{cases} \quad (12)$$

## 5 Proof of Main Theorems

Let  $T$  be the triangle of vertices  $(0, 0)$ ,  $(1, 0)$  and  $(0, 1)$ . We recall that  $\mathcal{S}_j$  is the collection of half open  $r$ -stair polygons  $S$  contained in  $T$  which  $r \leq 2j - 1$  and  $S$  is an exact  $j$ -fold tile, and  $\mathcal{S}^j$  is the collection of half open  $r$ -stair polygons  $S$  such that  $\text{Int}(T) \subset S$ ,  $r \leq 2j - 1$  and  $S$  is an exact  $j$ -fold tile. We denote by  $A_j$  the maximum area of polygons in  $\mathcal{S}_j$  and denote by  $A^j$  the minimum area of polygons in  $\mathcal{S}^j$ .

Let  $\mathcal{S}_j^*$  be the collection of half open  $r$ -stair polygons that contained in  $T$  and  $r \leq 2j - 1$ . Let  $\mathcal{S}_*^j$  be the collection of half open  $r$ -stair polygons that contain  $\text{Int}(T)$  and  $r \leq 2j - 1$ . Denote by  $A_j^*$  the maximum area of polygons in  $\mathcal{S}_j^*$ . Denote by  $A_*^j$  the minimum area of polygons in  $\mathcal{S}_*^j$ . By elementary calculations, one can obtain that  $A_j^* = \frac{j}{2j+1}$ , and  $A_*^j = \frac{2j+1}{4j}$ . Furthermore, we have the following lemmas.

**Lemma 5.1.** *Suppose that  $S \in \mathcal{S}_j^*$ . We have that  $|S| = \frac{j}{2j+1}$  if and only if  $S = \frac{1}{2j+1}S(j)$ .*

**Lemma 5.2.** *Suppose that  $S \in \mathcal{S}_*^j$ . We have that  $|S| = \frac{2j+1}{4j}$  if and only if  $S = \frac{1}{2j}S(j)$ .*

From the definitions, we obviously have  $\mathcal{S}_j \subset \mathcal{S}_j^*$  and  $\mathcal{S}^j \subset \mathcal{S}_*^j$ . Hence  $A_j \leq A_j^*$  and  $A^j \geq A_*^j$ . By Theorem 3.1, we know that  $\frac{1}{2j+1}S(j)$  and  $\frac{1}{2j}S(j)$  are also exact  $j$ -fold tiles. Therefore, from Lemma 5.2 and Lemma 5.1, we obtain

$$A_j = A_j^* = \frac{j}{2j+1}, \quad (13)$$

and

$$A^j = A_*^j = \frac{2j+1}{4j}. \quad (14)$$

Moreover, we have the following lemmas.

**Lemma 5.3.** *Suppose that  $S \in \mathcal{S}_j$ . We have that  $|S| = \frac{j}{2j+1}$  if and only if  $S = \frac{1}{2j+1}S(j)$ .*

**Lemma 5.4.** *Suppose that  $S \in \mathcal{S}^j$ . We have that  $|S| = \frac{2j+1}{4j}$  if and only if  $S = \frac{1}{2j}S(j)$ .*

From (6), (7), (13) and (14), one can obtain Theorem 1.2. We now suppose that  $T + \Lambda$  is a  $j$ -fold lattice covering of  $\mathbb{R}^2$ . By the definition and properties of  $S_j(\Lambda)$ , it is clear that  $S_j(\Lambda) \in \mathcal{S}_j$  and the density of  $T + \Lambda$  is  $\frac{|T|}{d(\Lambda)} = \frac{j|T|}{|S_j(\Lambda)|}$ . By Lemma 5.3, we have that the density of  $T + \Lambda$  is equal to  $\vartheta_L^j(T) = \frac{j|T|}{A_j}$  if and only if  $S_j(\Lambda) = \frac{1}{2j+1}S(j)$ . Note that  $S_j(\Lambda) + \Lambda$  is an exact  $j$ -fold lattice tiling of  $\mathbb{R}^2$ . This implies that the density of  $T + \Lambda$  is equal to  $\vartheta_L^j(T)$  if and only if  $\frac{1}{2j+1}S(j) + \Lambda$  is an exact  $j$ -fold lattice tiling of  $\mathbb{R}^2$ . From this and Theorem 3.1, one can obtain Theorem 1.4.

Suppose that  $T + \Lambda$  is a  $j$ -fold lattice packing of  $\mathbb{R}^2$ . By the definition and properties of  $S_j(\Lambda)$ , we have that  $S_j(\Lambda) \in \mathcal{S}^j$  and the density of  $T + \Lambda$  is  $\frac{|T|}{d(\Lambda)} = \frac{j|T|}{|S_j(\Lambda)|}$ . By Lemma 5.4, we know that the density of  $T + \Lambda$  is equal to  $\delta_L^j(T) = \frac{j|T|}{A_j}$  if and only if  $S_j(\Lambda) = \frac{1}{2j}S(j)$ . This can be deduced that the density of  $T + \Lambda$  is equal to  $\delta_L^j(T)$  if and only if  $\frac{1}{2j}S(j) + \Lambda$  is an exact  $j$ -fold lattice tiling of  $\mathbb{R}^2$ . From this and Theorem 3.1, one can obtain Theorem 1.3.

Finally, one can easily show that when  $m, n \in \{1, 2, \dots, 2j+1\}$  and  $m \neq n$ , we have  $\Lambda(m, j) \neq \Lambda(n, j)$ . Hence, Corollary 1.5 directly follows from Theorem 1.3, Theorem 1.4 and Theorem 4.4.

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